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NOTE

On Rational Interpolation to $|x|$ at the Adjusted Chebyshev Nodes

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Recently Brutman and Passow considered Newman-type rational interpolation to $|x|$ induced by arbitrary sets of symmetric nodes in $[-1, 1]$ and showed that under mild restrictions on the location of the interpolation nodes, the corresponding sequence of rational interpolants converges to $|x|$. They also studied the special case where the interpolation nodes are the roots of the Chebyshev polynomials and proved that for this case the exact order of approximation is $O(1/n \log n)$, which, in view of Werner's result, is the same as for rational interpolation at equidistant nodes. In the present note we consider the set of interpolation nodes obtained by adjusting the Chebyshev roots to the interval $[0, 1]$ and then extending this set to $[-1, 1]$ in a symmetric way. We show that this procedure improves the quality of approximation, namely we prove that in this case the exact order of approximation is $O(1/n^2)$. © 1998 Academic Press

1. INTRODUCTION

The function $|x|$ has been the focus of much research in approximation theory over the years. Its fundamental role in polynomial approximation is well illustrated by Lebesgue's proof of the Weierstrass approximation theorem, which is based solely on the fact that a single function $|x|$ can be approximated. However, as was shown by Bernstein [1], the order of the best uniform approximation of $|x|$ by *polynomials* is only $O(n^{-1})$.

In contrast to this, Newman [4] demonstrated that *rational* approximation to $|x|$ is much more favorable, namely $|x|$ may be approximated uniformly by rational functions at an exponential rate. Newman's result generated a great deal of research, much of which focused on the problem of sharpening the asymptotic results for the error in the best rational approximation. The most recent result in this direction is the proof of the

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so-called “8” conjecture by Stahl. (See [6], where the main result is presented and an extensive historical review is given.)

In [2] Brutman and Passow considered Newman-type rational approximation induced by arbitrary sets of interpolation points. Let $X=\{0<x_1^{(n)}<x_2^{(n)}<\cdots<x_n^{(n)}\leqslant 1\}$ be a set of n distinct points in $(0, 1]$ and let $p(x)=\prod_{k=1}^{n}(x+x_k^{(n)})$. (In the sequel, when there is no possibility for confusion, the superscript (n) will be omitted.) The rational function, corresponding to the set X is defined by

$$r_n(X; x)=x\,\frac{p(x)-p(-x)}{p(x)+p(-x)}.$$

It can be easily verified that $r_n(X; x)$ interpolates $|x|$ at the following set of $2n+1$ points: $\{-x_n, ..., -x_1, 0, x_1, ..., x_n\}$. Since the $r_n(X; x)$ as well as $|x|$ are even functions, the study of the approximation error $e_n(X; x)=|x|-r_n(X; x)$ may be restricted to the interval $[0, 1]$, where it can be represented in the form

$$e_n(X; x)=\frac{2xh_n(X; x)}{1+h_n(X; x)}, \qquad 0\leqslant x\leqslant 1, \tag{1}$$

where

$$h_n(X; x)=\frac{p(-x)}{p(x)}=\prod_{k=1}^{n}\frac{x_k-x}{x_k+x}. \tag{2}$$

In the sequel we will use the following general estimates which were proved in [2]:

Statement 1.1. Let $S_1=S_1^{(n)}(X)=\sum_{k=1}^{n} x_k^{(n)}$. Then

$$|e_n(X; x)|\leqslant\frac{2}{S_1}, \qquad -1\leqslant x\leqslant 1. \tag{3}$$

Statement 1.2. Let $A_n=A_n(X)=1/\sum_{k=1}^{n} x_k^{-1}$. Then

$$|e_n(X; x)|\leqslant 1/A_n, \qquad x\in[-x_1, x_1]. \tag{4}$$

In [3] Brutman and Passow studied the special case of interpolation nodes coinciding with the roots of the Chebyshev polynomial of even degree $T_{2n}(x)$, namely

$$X=\tilde{T}:=\{\cos((2k-1)\,\pi/(4n))\}_{k=1}^{n}.$$

They proved that the exact order of approximation of $|x|$ by $r_n(\tilde{T}; x)$ is $O(1/n \log n)$, which, in view of Werner's result [7], is the same as for rational interpolation at equidistant nodes.

In the present paper we consider the set of nodes obtained by adjusting the Chebyshev roots $\xi_k^{(n)} = \cos((2k-1)\,\pi/(2n))$, $k = 1, 2, ..., n$ to the interval $[0, 1]$, namely

$$X = T := \{x_k = \tfrac{1}{2}(1 + \xi_{n-k+1}^{(n)}) = \sin^2((2k-1)\pi/(4n))\}_{k=1}^{n}. \tag{5}$$

We show that this procedure improves the quality of approximation, namely we prove that in this case the exact order of approximation is $O(1/n^2)$.

Finally we would like to mention that the method of our proof is rather general and may be applied to other specific sets of interpolation points.

2. RESULTS

Consider the case of the adjusted Chebyshev nodes (5). Note first that since $S_1(T) = n/2$, the general formula (3) implies $|e_n(T; x)| \leqslant 4/n$. This estimate is rather conservative, as we will show. Our purpose is to find an exact order of approximation of $|x|$ by $r_n(T; x)$ and to this end we have to study thoroughly the behavior of the function $h_n(T; x)$. Since $|x|$ and $r_n(T; x)$ are even functions in $[-1, 1]$ we can restrict ourselves to $x \in [0, 1]$. For this interval, as can be easily verified, the function $h_n(T; x)$ may be represented in the form

$$h_n(T; x) = \frac{(-1)^n\, T_n(2x-1)}{T_n(2x+1)}, \qquad x \in [0, 1]. \tag{6}$$

The following estimate holds:

LEMMA 2.1. *For any $x \in [x_1, 1]$ and $n = 2, 3, ...$*

$$|h_n(T; x)| < \tfrac{1}{2}. \tag{7}$$

Proof. It follows from (6) that for $x \in [x_1, 1]$

$$|h_n(T; x)| \leqslant \frac{1}{T_n(2x+1)} \leqslant \frac{1}{T_n(2x_1+1)} := \frac{1}{T_n(\alpha_n)}, \tag{8}$$

where

$$\alpha_n := 2x_1 + 1 = 2 - \cos\frac{\pi}{2n}.$$

In the sequel we will use the following well-known representation of the Chebyshev polynomials [5]:

$$T_n(x) = \tfrac{1}{2}[(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n]. \tag{9}$$

Then from (8) we obtain

$$\begin{aligned} |h_n(T; x)| &\leqslant \frac{2}{(\alpha_n + \sqrt{\alpha_n{}^2 - 1})^n + (\alpha_n - \sqrt{\alpha_n{}^2 - 1})^n} \\ &< \frac{2}{(\alpha_n + \sqrt{\alpha_n{}^2 - 1})^n}. \end{aligned} \tag{10}$$

We want to show next that the sequence $\{D_n\}_{n=1}^{\infty}$ defined by

$$D_n := (\alpha_n + \sqrt{\alpha_n{}^2 - 1})^n, \qquad n = 1, 2, ...,$$

is strictly monotone increasing. To this end we put $x = \pi/2n$ and prove that

$$D(x) = [\alpha(x) + \sqrt{\alpha^2(x) - 1}]^{\pi/2x}, \qquad \text{where} \quad \alpha(x) = 2 - \cos x,$$

is a decreasing function of x for $x \in (0, \pi/2]$.

Since $D(x) > 0$, it suffices to verify that the logarithmic derivative of $D(x)$ is negative, namely that

$$\frac{D'(x)}{D(x)} = -\frac{\pi}{2x^2} \ln[\alpha(x) + \sqrt{\alpha^2(x) - 1}] + \frac{\pi\alpha'(x)}{2x\sqrt{\alpha^2(x) - 1}} < 0, \qquad x \in (0, \pi/2],$$

which is equivalent to proving that

$$L(x) := -\ln[\alpha(x) + \sqrt{\alpha^2(x) - 1}] + x\frac{\alpha'(x)}{\sqrt{\alpha^2(x) - 1}} < 0, \qquad x \in (0, \pi/2]. \tag{11}$$

But $L(0) = \lim_{x \to 0} L(x) = 0$ and therefore in order to prove (11) it suffices to verify that the following inequality holds:

$$L'(x) = x\left[\frac{\alpha'(x)}{\sqrt{\alpha^2(x) - 1}}\right]' < 0, \qquad x \in (0, \pi/2].$$

This last inequality is equivalent to the following, obviously valid inequality

$$\alpha''(x)[\alpha^2(x) - 1] - \alpha(x)[\alpha'(x)]^2 = (-2)[1 - \cos x]^2 < 0, \qquad x \in (0, \pi/2].$$

This proves the monotonicity of the sequence D_n. It remains to note that $D_2 = 4.4622...$ and the result follows from (10). ∎

Now we are in a position to prove our main result.

THEOREM 2.2. *For any $x \in [-1, 1]$ and $n = 2, 3, ...$ the following estimate holds*:

$$|e_n(T; x)| < \frac{8}{e^2(n^2-1)}. \tag{12}$$

Proof. As before we can restrict our analysis to $x \geqslant 0$. Consider first the case $x \in [0, x_1] = [0, \sin^2(\pi/(4n))]$. By applying the general estimate (4) and taking into account the following identity (see, e.g., [5])

$$A_n(T) := \sum_{k=1}^{n} \frac{1}{x_k} = \sum_{k=1}^{n} \frac{1}{\sin^2((2k-1)/4n)\pi} = 2n^2,$$

we get

$$|e_n(T, x)| \leqslant \frac{1}{2n^2}, \qquad x \in [0, x_1]. \tag{13}$$

Now let us consider the error of approximation in the interval $[x_1, 1]$. Applying (1), (6), and the lemma we obtain for $n \geqslant 2$

$$|e_n(T; x)| \leqslant \frac{2x\,|h_n(T; x)|}{1-|h_n(T; x)|} < 4x\,|h_n(T; x)| = \frac{4x\,|T_n(2x-1)|}{T_n(2x+1)} \leqslant \frac{4x}{T_n(2x+1)}. \tag{14}$$

Let $F_n(x) := x/T_n(2x+1)$. Then by applying the transformation $t = 2x+1$ and using once again the representation (9), we find

$$\max_{x_1 \leqslant x \leqslant 1} F_n(x) = \max_{\alpha_n \leqslant t \leqslant 3} \frac{t-1}{2T_n(t)} < \max_{\alpha_n \leqslant t \leqslant 3} \frac{t-1}{(t+\sqrt{t^2-1})^n} := \max_{\alpha_n \leqslant t \leqslant 3} G_n(t). \tag{15}$$

An easy computation reveals that for $n \geqslant 2$ the function $G_n(t)$ attains its maximal value on the interval $[\alpha_n, 3]$ at the point $t_{\max} = \beta_n := 1 + 2/(n^2-1)$. Moreover, by applying the method used in proving the monotonicity of the sequence D_n in Lemma 2.1, one can verify that the sequence $(\beta_n + \sqrt{\beta_n^2 - 1})^n\}_{n=2}^{\infty}$ monotonically decreases to e^2. Therefore we have

$$\max_{\alpha_n \leqslant t \leqslant 3} G_n(t) = G_n(\beta_n) < \frac{2}{e^2(n^2-1)}, \qquad n = 2, 3, ... \tag{16}$$

Combining (14)–(16) yields

$$|e_n(T;x)| < \frac{8}{e^2(n^2-1)}, \qquad x \in [x_1, 1], \quad n = 2, 3, ... \tag{17}$$

Comparison of (13) and (17) completes the proof of the theorem. ∎

Finally we show that the estimate (12) is sharp, namely the following result holds:

THEOREM 2.3. *Let* $x^* = 1/(4n^2)$. *Then*

$$|e_n(T;x^*)| \geqslant \frac{C}{n^2}, \qquad n \geqslant n_0. \tag{18}$$

Proof. Note first that for $n \geqslant 1$, $x^* \in [0, x_1]$ and since in this interval $0 \leqslant h_n(T;x) \leqslant 1$, we can write

$$4n^2\,|e_n(T;x^*)| = \frac{e_n(T;x^*)}{x^*} = \frac{2h_n(T;x^*)}{1+h_n(T;x^*)} \geqslant h_n(T;x^*).$$

Thus in order to prove (18) we have to show that the sequence $\{R_n\}_{n=1}^{\infty}$ defined by

$$R_n := \frac{1}{h_n(T;x^*)} = \frac{T_n(2x^*+1)}{(-1)^n\,T_n(2x^*-1)} := \frac{P_n}{Q_n} \tag{19}$$

is bounded. To this end note first that, as may be easily verified,

$$T_n(2x-1) = T_{2n}(\sqrt{x}), \qquad x \geqslant 0,$$

and thus

$$Q_n = (-1)^n\,T_{2n}(\sqrt{x^*}) = (-1)^n \cos\left(2n \arccos\frac{1}{2n}\right) \to \cos 1, \qquad \text{as} \quad n \to \infty.$$

Therefore it suffices to consider the behavior of the numerator of (19), which in view of (9) may be represented in the form

$$P_n = \tfrac{1}{2}[(t^* + \sqrt{(t^*)^2-1})^n + (t^* + \sqrt{(t^*)^2-1})^n],$$

where $t^* = 2x^*+1 = 1 + 1/2n^2$. A routine computation yields

$$\lim_{n\to\infty} P_n = \frac{1}{2}\left(e + \frac{1}{e}\right),$$

and the result follows. ∎

REFERENCES

1. S. Bernstein, Sur la meilleure approximation de $|x|$ par des polynômes de degrés donnés, *Acta Math.* **37** (1913), 1–57.
2. L. Brutman and E. Passow, On rational interpolation to $|x|$, *Constr. Approx.* **13** (1997), 381–391.
3. L. Brutman and E. Passow, Rational interpolation to $|x|$ at the Chebyshev nodes, *Bull. Austral. Math. Soc.* **56** (1997), 81–86.
4. D. Newman, Rational approximation to $|x|$, *Michigan Math. J.* **11** (1964), 11–14.
5. T. J. Rivlin, "Chebyshev Polynomials," 2nd ed., Wiley, New York, 1990.
6. H. Stahl, Best uniform rational approximation of $|x|$ on $[-1, 1]$, *Mat. Sb.* **183** (1992), 85–118.
7. H. Werner, Rationale Interpolation von $|x|$ in äquidistanten Punkten, *Math. Z.* **180** (1982), 85–118.

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